Hopf term induced by fermions

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Abstract

We derive an effective action for Dirac fermions coupled to O(3) non-linear σ-model (NLσM) through the Yukawa-type interaction. The nonperturbative (global) quantum anomaly of this model results in a Hopf term for the effective NLσM. We obtain this term using the "embedding" of the CP^1 model into the CP^{M} generalization of the model which makes the quantum anomaly perturbative. This perturbative anomaly is calculated by means of a gradient expansion of a fermionic determinant and is given by the Chern-Simons term for an auxiliary gauge field.

Key words: quantum anomalies, fermionic determinant, Hopf term, effective action

It is well-known that nonperturbative anomalies[1] in gauge theories can be reduced to perturbative ones by embedding the gauge group into a bigger one[2,3]. In this paper we show that a similar method can be used to calculate the effect of nonperturbative anomalies on the effective action of a non-linear σ model induced by fermions.

We consider an effective action $S_{eff}(n)$ of Dirac fermions on the three-dimensional sphere S^3 coupled to a background chiral field n:

$$e^{-S_{eff}(n)} = \int d\psi\, d\bar{\psi}\, \exp\left(-\int\limits_{S^3} d^3x\ \bar{\psi}\left[i\partial\!\!\!/ + im\hat{n}\right]\psi\right). \qquad (1)$$

Here $\partial\!\!\!/ = \gamma^\mu\partial_\mu$ where the γ^μ are three-dimensional gamma-matrices which can be chosen, e.g., to be Pauli matrices, $\hat{n} = \vec{n}\cdot\vec{\tau}$ with $\vec{n} \in S^2$, $\vec{n}^2 = 1$, $\vec{\tau}$ is a set of Pauli matrices acting in the isospace, and we use a Euclidian formulation.

We calculate the variation of the effective action $S_{eff}(n) = -\ln\det D$, $D = i\partial\!\!\!/ + im\hat{n}$ with respect to n.

$$\delta S_{eff} = -\mathrm{Tr}\left[\delta D D^\dagger (DD^\dagger)^{-1}\right], \qquad (2)$$

where $\delta D = im\delta\hat{n}$ and $D^\dagger = i\partial\!\!\!/ - im\hat{n}$. Then we use[4] $DD^\dagger = -\partial^2 + m^2 + m\partial\!\!\!/\hat{n}$ and expand in $m\partial\!\!\!/\hat{n}$ obtaining $(DD^\dagger)^{-1} = G_0 - G_0 m\partial\!\!\!/\hat{n}G_0 + G_0(-m\partial\!\!\!/\hat{n}G_0)^2 + \ldots$, where $G_0 = \frac{1}{-\partial^2+m^2}$.

Calculating traces and leaving only first nonzero orders in $1/m$ of real and imaginary parts of the effective action we obtain (only trace in isospace is left)

$$\delta S_{eff} = \delta S_{Re} + \delta S_{Im}, \tag{3}$$

$$\delta S_{Re} = \frac{|m|}{8\pi}\int d^3x\, \mathrm{tr}\,(\partial_\mu\delta\hat{n})(\partial_\mu\hat{n}) + \ldots, \tag{4}$$

$$\delta S_{Im} = -i\frac{\mathrm{sgn}(m)}{32\pi}\int d^3x\, \epsilon^{\mu\nu\lambda}\,\mathrm{tr}\,(\hat{n}\delta\hat{n}\partial_\mu\hat{n}\partial_\nu\hat{n}\partial_\lambda\hat{n}). \tag{5}$$

Now our goal is to restore the effective action from its variation (3-5). For the real part we have

$$S_{Re} = \frac{|m|}{16\pi}\int d^3x\, \mathrm{tr}\,(\partial_\mu\hat{n})^2 + \ldots. \tag{6}$$

It is straightforward to check that the variation of the imaginary part of the action (5) is identically zero and naively $S_{Im} = 0$. However, because of the nontrivial homotopy group $\pi_3(S^2) = Z$, the configurations of n are divided into topological classes[5] labeled by an integer-valued Hopf invariant $H(n)$. If $S_{Im} \sim H(n)$ then $\delta S_{Im} = 0$ and we can not find the "Hopf term" from our perturbative calculation. The presence of the Hopf term in the effective action is of great importance because it changes spin and statistics of solitons of NLσM[6].

In the following we show that the correct result for the imaginary part is

$$S_{Im} = -i\pi\,\mathrm{sgn}(m)\,H(n). \tag{7}$$

The value of the coefficient in front of the Hopf invariant corresponds to the Fermi-Dirac statistics of solitons which agrees with their fermionic charge[7,4].

To find the imaginary part of the effective action we generalize the model (1), changing the size of isospace and replacing $\hat{n}$ by $\hat{n} = 2zz^\dagger - 1$ with $z^t = (z_1, z_2, \ldots, z_{M+1})$ – complex vector with unit modulus $z^\dagger z = 1$. The $(M+1)\times(M+1)$ matrix $\hat{n}$ does not depend on the phase of z, i.e., z should be considered as a CP$^{\mathrm{M}}$ field[8]. We note that for the particular configuration $z^t = (z_1, z_2, 0, \ldots, 0)$ the fermions with isospace indices higher than 2 are decoupled from z and do not contribute z-dependent terms into the effective action. As a consequence we can obtain the effective action for (1) by

restricting the effective action of the $\mathrm{CP^M}$ model to particular configurations $z^t = (z_1, z_2, 0, \ldots, 0)$.

It is easy to check that $\hat{n}^2 = (2zz^\dagger - 1)^2 = 1$ and perturbative results (3-5) are still valid. However, for $M > 1$ the homotopy group $\pi_3(\mathrm{CP^M}) = 0$ and there are no topologically nontrivial configurations of $\hat{n}$. The effective action can be found perturbatively! Substituting $\hat{n} = 2zz^\dagger - 1$ into (5) we obtain after some algebra

$$\delta S_{Im} = i\frac{\mathrm{sgn}(m)}{2\pi} \int d^3x \, \epsilon^{\mu\nu\lambda} \, \delta a_\mu \, \partial_\nu a_\lambda, \tag{8}$$

where $a_\mu = z^\dagger(-i\partial_\mu)z$. From here we obtain

$$S_{Im} = i\frac{\mathrm{sgn}(m)}{4\pi} \int d^3x \, \epsilon^{\mu\nu\lambda} \, a_\mu \, \partial_\nu a_\lambda. \tag{9}$$

Restricting (9) to particular $\mathrm{CP}^1 = S^2$ configurations we have (7) with the well-known expression for the Hopf invariant[5]

$$H(n) = -\frac{1}{4\pi^2} \int d^3x \, \epsilon^{\mu\nu\lambda} \, a_\mu \, \partial_\nu a_\lambda, \tag{10}$$

where $a_\mu = z^\dagger(-i\partial_\mu)z$ with a two-component z and $\vec{n} = z^\dagger \vec{\tau} z$.

Combining (6) and (7) we obtain for the effective action of the $\mathrm{CP^M}$ non-linear σ-model induced by Dirac fermions

$$S_{eff} = \int d^3x \left\{ \frac{|m|}{16\pi} \, \mathrm{tr} \, (\partial_\mu \hat{n})^2 + i\frac{\mathrm{sgn}(m)}{4\pi} \, \epsilon^{\mu\nu\lambda} \, a_\mu \, \partial_\nu a_\lambda \right\}, \tag{11}$$

where we kept only the imaginary part of the action and the terms of order m in the real part. The second term of (11) is a perturbative "Chern-Simons" term in the case of $M > 1$. It becomes a nonperturbative (global) Hopf term in the case of $M = 1$.

In conclusion, we have derived the effective action for the $\mathrm{CP^M}$ non-linear σ-model induced by Dirac fermions on a three-dimensional sphere. We have shown that this effective action has a nontrivial topological term which (for $M = 1$) is equal to the well-known Hopf term for the O(3) non-linear σ-model. We used the method of embedding[2,3] known for global anomalies in gauge theories.

We would like to point out some differences between global terms for the σ-model and for the gauge field models. In the case of gauge fields there al-

ways exists a direct interpolation between configurations which differ by nontrivial gauge transformation[1]. Therefore, the question of the relative phase of fermionic determinants for those configurations is well-defined. In the case of the NLσM there is no such direct interpolation between configurations from different topological classes. Therefore, strictly speaking, these topological classes can be weighed in the partition function corresponding to (1) with arbitrary weights. The imaginary part of an effective action, or the relative phase of determinants with chiral field configurations belonging to different topological classes, is not defined. However, in a realistic physical model there might be some regularization which allows one to connect field configurations from different topological classes. E.g., if a theory is defined on a lattice, the singular processes changing topological classes are allowed. Another possibility is to make a constraint, $\hat{n}^2 = 1$, soft. Then at some points in space-time $\hat{n} = 0$, the target manifold is not S^2, and there are no distinct topological classes anymore.

In this sense the "embedding method" we used is not just a technical trick but is essentially a method of regularization which allows us to interpolate between different topological classes of NLσM.

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References

[1] E. Witten, Phys. Lett. **117B** (1982) 324-328
An SU(2) Anomaly

[2] S. Elitzur and V.P. Nair, Nucl. Phys. **B243** (1984) 205-211
Nonperturbative Anomalies in Higher Dimensions

[3] F.R. Klinkhamer, Phys. Lett. **256B** (1991) 41-42
Another look at the SU(2) anomaly

[4] A.G. Abanov and P.B. Wiegmann, Nucl. Phys. **B570** (2000) 685-698
Theta-terms in non-linear sigma-models.

[5] B.A. Dubrovin, A.T. Fomenko, S.P. Novikov,
Modern Geometry-Methods and Applications : Part II, the Geometry and Topology of Manifolds (Graduate Texts in Mathematics, Vol 104), Springer-Verlag, 1985.

[6] F. Wilczek and A. Zee, Phys. Rev. Lett. **51** (1983) 2250-2252
Linking Numbers, Spin, and Statistics of Solitons

[7] T. Jaroszewicz, Phys. Lett. **B193** (1987) 479-485
Fermion-Induced Spin of Solitons: Vacuum and Collective Aspects

[8] The idea of using CP^{M} representation to derive Hopf term belongs to T. Jaroszewicz[9]. However, he used "gauge rotation" with nonzero Jacobian and the derivation of Ref.[9] is incorrect.

[9] T. Jaroszewicz, Phys. Lett. **B159** (1985) 299-302
Induced Topological Terms, Spin and Statistics in (2+1) Dimensions